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TITLE

Efficient stress resultant plasticity formulation for thin shell applications: Implementation and numerical tests

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ABSTRACT

In the following an implementation of a stress resultant plasticity model with a triangular flat shell finite element is described. The element uses 4 integration points in the plane (18 degrees-of-freedom, three displacements and rotations at the corner nodes), no integration over thickness, and small strain/large rotation formulation. The element field interpolation consists of higher order assumed strain terms which gives very good elastic performance, and is based on the free formulation methodology. A true tangent stiffness is employed in the Newton-Raphson equilibrium iterations, i.e. consistency with respect to stress resultant backward Euler updating (plasticity) and large rotations. Hence quadratic rate of convergence is obtained. Several test cases are analysed, some of them show extremely nonlinear behavoiur. The response is compared to corresponding results in the literature, showing acceptable/good performance.

KEYWORDS	ENGLISH	NORWEGIAN
GROUP 1	Numerical methods	Numeriske metoder
GROUP 2	Shell structures	Skallkonstruksjoner
SELECTED BY AUTHOR	Plasticity	Plastisitet
	Finite element method	Element metode

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1. INTRODUCTION

Shell type of structures occur frequently in offshore applications. For instance a steel jacket consists of tubular members that may be regarded as beam-columns in many situations, but in some special situations due to e.g. accidental/extreme loading, some of the members should be analysed by means of shell theory. It should be noted that often some few structural components are critical with respect to global load carrying capability. If one wish to account for local effects such as dents/local buckling and variable temperature fields in members and surface cracks in tubular joints in a more detailed manner, these components may be modelled as shell element based substructures in a global model based on beam-column theory (e.g. USFOS). With this, accurate account of load redistribution and dent/damage growth is obtained.

There exist many shell finite elements with different levels of sophistication. Some points that need to be addressed in the process of chosing an element are:

1. thin or thick shell theory (i.e. is out-of-plane shear deformations of importance when compared to the bending deformations);
2. for elasto-plastic conditions: is a layer approach (integration points through thickness) or a stress resultant approach best?
3. small/large strains versus small/large rotations (large strains would require updating of shell thickness);
4. type of element interpolation (triangular or rectangular finite element, number of integration points (selective reduced or full integration, assumed strain interpolation));
5. efficiency (related to using an approach that accounts for special features of the application versus using a general (perhaps slower) approach).

Regarding point 1), one knows from bending of beams that if length to height ratios are larger than 5-10, the deformation is bending dominated. For a cylindrical shell loaded transversally (radially) this condition could roughly be $D/t > 10$. Hence, in many cases a thin shell theory is sufficient. Regarding point 2), if first fiber yielding is important, a layer approach is required. Furthermore, inelastic buckling of a shell depends on stress distribution over shell thickness. A stress resultant approach may be crude if accurate simulation of shell collapse is to be achieved. In a shell with statically indeterminate stress conditions (typical), for bending dominated loading (i.e. load transversal to shell wall), the stress resultant approach may be sufficient for simulating the redistribution of stress. Regarding 3), large strain is important in metal forming, whereas for collapse of cylindrical shells, the situation in the regions with the largest geometry change is described by small/moderate strains but large rotations. This observation is important, as significant simplications in finite element formulation may be utilized. With respect to points 4) and 5), nonlinear shell fea is time consuming, and one is motivated to employ as simple elements as possible (low order) with as few integration points as possible.

In the following an implementation of a stress resultant plasticity model with a triangular flat shell finite element is described. The element uses 4 integration points in the plane (18 degrees-of-freedom, three displacements and rotations at the corner nodes), no integration over thickness, and small strain/large rotation formulation. The element field interpolation consists of higher order assumed strain terms which gives very good elastic performance, and is based on the free formulation methodology (Bergan, Nygård, Felippa). Details of the consistent co-rotated finite element formulation for linear material is given in the thesis by B.Haugen (1994).

2. ELASTO-PLASTIC FORMULATION

The relationship between stress and elastic strain for thin shells is described by

$$\sigma = \left[\sigma_{xx}, \sigma_{yy}, \tau_{xy}\right]^T$$
$$\varepsilon^e = \left[\varepsilon^e_{xx}, \varepsilon^e_{yy}, \gamma^e_{xy}\right]$$
$$\sigma = \mathbf{D}\,\varepsilon^e \qquad \mathbf{D} = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Closed form for stress resultants is obtained by integrating over thickness.

$$\mathbf{N} = \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_t \sigma dz = t\,\mathbf{D}\,\mathbf{B}_o\,\mathbf{v}_o \qquad \mathbf{M} = \begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} = -\int_t \sigma \cdot z\,dz = \int_t \mathbf{D}\,\mathbf{B}_b\,z^2\,dz\,\mathbf{v}_b$$

The stress resultant yield criterion by Ilyushin is utilized herein:

$$Q_t = \frac{\overline{N}}{N_o^2}, \qquad \overline{N} = (N_x^2 + N_y^2 - N_x\,N_y + 3N_{xy}^2), \qquad N_o = \sigma_o t$$
$$Q_m = \frac{\overline{M}}{M_o^2}, \qquad \overline{M} = (M_x^2 + M_y^2 - M_x\,M_y + 3M_{xy}^2), \qquad M_o = \frac{1}{4}\sigma_o t^2$$
$$Q_{tm} = \frac{\overline{P}}{N_o M_o}, \quad P = (N_x M_x + N_y\,M_y - \frac{1}{2}N_x\,M_y - \frac{1}{2}N_y\,M_x + 3N_{xy}\,M_{xy})$$

$$f(\mathbf{N},\mathbf{M}) = \left(\frac{\overline{N}}{t^2} + \frac{4\ s\ P}{t^3\ \sqrt{3}} + \frac{16\overline{M}}{t^4}\right)^{1/2} - \sigma_o = 0$$
$$s = P/|P| \in \{-1,+1\}$$

An alternative quadratic form of f is discussed subsequently. The backward Euler (BE) stress resultant update and its consistent linearization are outlined. Isotropic hardening is assumed as hardening model due to its simplicity. It should be noted that due to s, the yield surface has corners, i.e. discontinuous gradients. This needs special treatment.

Denoting the current plastic membrane strain and curvature increment from global load level *n* to *n+1* $\Delta\varepsilon^P = [\Delta\varepsilon_u^P, \Delta\,\kappa^P\,]^T$, the BE stress resultants by $\sigma_C = [\boldsymbol{N}_C\,, \boldsymbol{M}_C]^T$, and the elastic predictor stress resultants by σ_B, the update is obtained as follows:

$$\Delta\,\varepsilon_C^P = \Delta\lambda_C \left.\frac{\partial f}{\partial\sigma}\right|_C = \Delta\lambda_C\,\mathbf{a}_C$$

$$\Rightarrow \sigma_C = \sigma_B - \Delta\lambda\;\mathbf{C}\,\mathbf{a}_C,\quad \sigma_B = \mathbf{C}(\varepsilon_n + \Delta\varepsilon)$$

$$\mathbf{C} = \begin{bmatrix} t\mathbf{D} & \mathbf{0} \\ \mathbf{0} & \dfrac{t^3}{12}\mathbf{D} \end{bmatrix}$$

One disadvantage with Ilyushin yield surface is its corners. Figure 1 illustrates this. If return from elastic prediction at *B* is based on the current governing yield surface *f*(*s=-1*) (depicted in *M-N*-subspace), one ends at the wrong point *C*(*=n+1*). The erroneous *C* leads to violation of *g*(*s=1*). Algorithms handling such situations have been proposed by Simo and co-workers. However, few investigations of applications have been presented in the literature. Crisfield reports convergence problems with two active yield surfaces. Hence, alternatives may be attractive. A very simple modification of *f* and *g*, setting *s=0*, leads to a hypersphere as yield surface. In this case no corner problem occurs. Figure 2 shows the (non-conservative) error in yield condition for *M-N* case. In some instances such an error may be acceptable when compared to other sources of uncertainties in model parameters etc. It should be noted that for a shell element based superelement representing a critical member/component in a large redundant structure, this inconsistency may be acceptable.

Fig.1

Fig.2

An elegant and efficient way of dealing with the yield surface corners is derived by Matthies (1989) and employed for plates and shells by Ibrahimbegovic and Frey (1993a, b). For a single active yield surface an advantage is that only iterations on one scalar equation for the plastic multiplier $\Delta\lambda$ is necessary instead of the additional stress or plastic strain residual (two active surfaces $\Rightarrow$ two coupled scalar equations).

First, the yield condition is rewritten as

$$f = \boldsymbol{\sigma}^T\,\mathbf{A}\,\boldsymbol{\sigma} - \left(1 + \frac{E_p \varepsilon^p}{\sigma_y}\right)^2 = 0$$

$$\mathbf{A}\big|_s = \begin{bmatrix} \dfrac{1}{n_o^2}\overline{\mathbf{A}} & \dfrac{s}{2\sqrt{3}\,m_o n_o}\overline{\mathbf{A}} \\ \dfrac{s}{2\sqrt{3}\,m_o n_o}\overline{\mathbf{A}} & \dfrac{1}{m_o^2}\overline{\mathbf{A}} \end{bmatrix}$$

$$\overline{\mathbf{A}} = \begin{bmatrix} 1 & -\dfrac{1}{2} & 0 \\ -\dfrac{1}{2} & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$m_o = \frac{1}{4}\sigma_y t^2, n_o = \sigma_y t$$

The associated flow rule now reads

$$d\varepsilon^p = d\lambda 2\mathbf{A}\sigma$$
$$\sigma^T d\varepsilon^p = \sigma_o d\varepsilon^p \Rightarrow d\varepsilon^p = 2\sigma_o d\lambda$$

BE integration of the flow rule yields

$$\Delta\varepsilon^p = \Delta\lambda_{n+1} 2\mathbf{A}\sigma_{n+1}$$
$$\Rightarrow \sigma_{n+1} = \sigma_{trial} - \mathbf{C}\,\Delta\varepsilon^p$$
$$\sigma_{n+1} = \overline{\mathbf{Q}}^{-1}\sigma_{trial}, \quad \overline{\mathbf{Q}} = [\mathbf{I} + 2\Delta\lambda\,\mathbf{CA}]$$

Now $\boldsymbol{\sigma}_{n+1}$ depends only on $\Delta\lambda$, hence ε^p_{n+1} and $\boldsymbol{\varepsilon}^p_{n+1}$ also do. The discrete yield condition now is a function of $\Delta\lambda$ only.

Solving $f(\Delta\lambda_{n+1})$ the stress update follows directly from $\overline{\mathbf{Q}}$ and σ_{trial} . In order to obtain $f(\Delta\lambda_{n+1})$ some matrix manipulations is carried out on the $\overline{\mathbf{Q}}^{-1}$ matrix by means of diagonalization with eigenvalues and vectors.

$$(\mathbf{CA})\mathbf{E} = \mathbf{E}\Lambda \Rightarrow \mathbf{CA} = \mathbf{E}\Lambda\mathbf{E}^{-1}$$
$$\Rightarrow [\mathbf{I} + 2\Delta\lambda\,\mathbf{CA}]^{-1} = \mathbf{E}\left[\underbrace{\mathbf{I} + 2\Delta\lambda\,\Lambda}_{\mathbf{Q}_d^{-1}}\right]^{-1}\mathbf{E}^{-1}$$
$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 & . & . & 0 \\ 0 & \Lambda_2 & 0 & .. & 0 \\ 0 & & & & \\ . & & & & \\ . & & & & \Lambda_6 \end{bmatrix}$$
$$\Lambda = \Lambda_A\,\Lambda_C$$

Employing the matrix of eigenvectors and eigenvalues, ***E*** and ˇ in *f*, an explicit equation for $\Delta\lambda$ is obtained and is solved by Newton iterations.

$$f = \sigma^T_{trial}\,\overbrace{\mathbf{E}^{-T}\mathbf{Q}_d^{-1}\,\mathbf{E}^T\,\mathbf{AE}\,\overline{\mathbf{Q}}_d^{-1}\,\mathbf{E}^{-1}}^{\mathbf{A}^*}\,\sigma_{trial} - \left(1 + \frac{E_p}{\sigma_y}\left(\varepsilon^p_n + 2\Delta\lambda\sqrt{\sigma^T_{trial}\,\mathbf{A}^*\,\sigma_{trial}}\right)\right)^2 = 0$$

Note that for a given shell (***C***, *t*) the eigenvalue-calculation is only needed once, i.e. no computational expense at all.

The consistent tangent is obtained as follows

$$\varepsilon_{n+1} = \mathbf{C}^{-1}\sigma_{n+1} + \varepsilon_{n+1}^{p}$$
$$d\varepsilon_{n+1} = \mathbf{C}^{-1}d\sigma_{n+1} + d\varepsilon_{n+1}^{p}$$
$$= H^{-1}d\sigma + 2\mathbf{A}\sigma d\lambda$$

$$\Delta\varepsilon_{p} = 2\Delta\lambda\,\sigma, \quad \sigma = \sqrt{\sigma^{T}\mathbf{A}\,\sigma}$$
$$df_{n+1} = 0 \Rightarrow d\lambda = \frac{1}{\beta}\,2\sigma^{T}\mathbf{A}\,d\sigma$$
$$\beta = \frac{2\,\alpha\,\sigma}{1-\alpha\dfrac{\Delta\lambda}{\sigma}}, \quad \alpha = \frac{2E_{p}}{\sigma_{y}^{2}}\left(\sigma_{y} + E_{p}\varepsilon_{n+1}^{p}\right)$$

$$\Rightarrow \left[\mathbf{H}^{-1} + \frac{1}{\beta}\mathbf{g}\,\mathbf{g}^{T}\right]d\sigma = d\varepsilon$$
$$\mathbf{H}^{-1} = \mathbf{C}^{-1} + 2\Delta\lambda\;\mathbf{A}, \;\; \mathbf{g} = 2\mathbf{A}\sigma$$
$$\Rightarrow d\sigma = \left[\mathbf{H} - \frac{\mathbf{H}\,\mathbf{g}\,\mathbf{g}^{T}\,\mathbf{H}}{\mathbf{g}^{T}\,\mathbf{H}\,\mathbf{g} + \beta}\right]d\varepsilon \qquad \text{(from Sherman - Morrison formula)}$$
$$= \mathbf{C}_{t}\;d\,\varepsilon$$

A closed from expression for ***H*** is obtained from

$$\mathbf{H} = \mathbf{E}\left[\mathbf{I} + 2\Delta\lambda\,\Lambda\right]^{-1}\mathbf{E}^{-1}\,\mathbf{C}$$

If two yield surfaces are active (i.e. in a corner region) the above derivation is somewhat more complicated. Now we have $\boldsymbol{A}(s=1) = \boldsymbol{A}_1$ and $\boldsymbol{A}(s=1) = \boldsymbol{A}_2$, and we need to determine $d\lambda_1$ and $d\lambda_2$ from consistency conditions. The derivation is not given here, see Ibrahimbegovic and Frey (1993). It should be noted, however, that analogous closed form relationships as above are obtained.

In the following implementation $s=0$. Hence, the following matrices are applied:

$$\Lambda_{A} = diag\left[\frac{1}{2n_{o}^{2}}, \frac{3}{2n_{o}^{2}}, \frac{3}{n_{o}^{2}}, \frac{1}{2m_{o}^{2}}, \frac{3}{2m_{o}^{2}}, \frac{3}{m_{o}^{2}}\right]$$
$$\Lambda_{c} = diag\left[\frac{1}{1-\nu}, \frac{1}{1+\nu}, \frac{1}{2(1+\nu)}, \frac{t^{2}}{12(1-\nu)_{1}}, \frac{t^{2}}{12(1+\nu)}, \frac{t^{2}}{24(1+\nu)}\right]Et$$

$$\mathbf{E} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

3. NUMERICAL SIMULATIONS

3.1 Cantilever plate, pure bending

A simple test for pure bending behaviour is illustrated in Fig. 3. Here a plate fixed at one end is subjected to a bending moment at the other end. The dimensions of the plate is 100*40*10 (mm). The yield stress is 400 Mpa, the hardening was taken as zero. It is clearly seen that modelling the plate with 2 triangular elements leads to too rigid behaviour both in the elastic and plastic regime. The model with 10 triangular elements approaches the exact plastic capacity (=1). Interestingly, one observes that although the cantilever has ideally a constant curvature, the locations of the integration points (midsides and centroid of the triangles) lead to an inaccurate distribution of curvature in the plate, Further mesh refinement will improve this situation.

Fig. 3

3.2 Cantilever plate, pure axial load

Fig. 4 depicts the axial load versus axial elongation for the plate. The plastic capacity is 4000. A very small hardening is used in this simulation. The number of elements is 6. Comparing with the performance of the elements in constant curvature, the performance is better when the elements is subjected to a constant membrane deformation.

Fig.4

3.3 Plate with out-of plane line load, accounting for membrane force

The load is applied at midlength of the plate. Two of the opposite boundaries are fixed with respect to in-plane motion and free to rotate, the two other opposite boundaries are free. The internal forces in the plate goes from a pure bending dominated situation to a purely membrane dominated situation. 12 triangular elements are employed in the simulation. The material is nonhardeing with yield stress 400 Mpa. It is seen from Fig. 5 that this mesh is to coarse to capture the plastic bending moment capacity, i.e. an overprediction (cfr. section 3.1). When the membrane situation takes over, the simulation agrees well with the analytic, rigid plastic solution.

Fig.5

3.4 Plate with uniformly distributed transversal load, acc. for membrane force

Half of a plate with the same material and boundary conditions as in the previous section is modeled with 8 triangular elements. Fig. 6 illustrate a reasonable correspondence with the analytic rigid plastic solution.

Fig.6

3.5 Rectangular plate simply supported on all edges, subjected to uniform distributed loading

The plate has a length to width ratio of 3. There exist both lower and upper bound plastic capacity solutions to this problem. Fig. 7 shows these very close analytic solutions along with two simulations. One uses 48*2 elements (whole plate), the other 200*2 elements. The convergence to an analytic solution is observed. This case is a good test for the bending part of the yield condition. For the deflection levels plotted, the contribution of nonlinear geometrical terms is negligible.

Fig.7

3.6 Collapse of a simply supported plate with imperfection, subjected to axial loading

The same plate geometry as in the previous section is used along with the same boundary conditions. The example is analyzed in the thesis by T.Søreide(1977). The yield stress is 320Mpa, the hardening modulus was taken as 3500Mpa (this is a slight simplification to the real material curve). An imperfection with maximum of 0.5 thickness at plate center was employed. The imperfection shape is one sinusoidal half wave in each direction. This gives a very interesting response, because the first buckling mode for such a plate (length to width 3) is three half-waves in the longest direction. The switch from one to three waves as the axial load is incrased is plotted in Fig. 8. The correspondence with Søreide's results is good. The axial load versus axial displacement is shown in Fig.9. Some effect of mesh refinement is observed, but the performance of the 48*2 mesh is quite good. Fig. 10 illustrates the deformed mesh before and after the switch from one to three half-waves for the fine mesh.

Fig.8

Fig.9

Fig.10

3.7 Collapse analysis of the Scordelis-Lo roof

This case is an interesting problem, where the effect of combined membrane and bending stress resultants in the yield condition (cfr. Fig.2) is examined. Furthermore it is a case showing very nonlinear behaviour. The roof has cylindrical shape, is simply supported at the two opposite curved edges, the two remaining straight edges are free. Then the analysis increments the self-weight of the roof until it collapses. Fig.11 illustrate the calculated response. One quarter of the roof is modelled. In the investigation by Peric and Owen (1991) a layer approach is used. Hence, an accurate description of stresses over shell thickness is obtained. As the simplified yield surface employed in the present study is nonconservative, the comparison with a layer approach is interesting. The curves denoted 16*16 correspond to 16*16*2 triangular elements etc. It is noted that the overall correspondence between the two approaches is very good. But reducing the yield stress by 12% according to the nonconservatism in the yield surface, one sees that the maximum load is close the the one obtained with the layer approach (see curves 8*8 and 8*8*0.88).
Fig.12 a-b-c show the initial geometry, intermediate deformed geometry (where the top of the roof actually moves upwards), and the configuration after the roof starts to collapse (i.e. the top of the roof moves downwards).

Fig.11

Fig.12 a-b

Fig.12 c

3.8 Pinched cylinder

The pinched cylinder shown in Fig.13a is analysed with several mesh refinements, see Fig. 13c. Comparing with the published results from Brank et al, (1997) and Simo and Kennedy (1992) shows very good correspondence. This case represents a complex shell stress distribution, with nonproportional membrane and bending moment histories. The coarse meshes show nonphysical mesh dependent snap troughs, whereas the fine mesh exhibit the correct physical behaviour, with one switch in deformation mode at a displacement of 150.

Fig.13

4. REFERENCES

Bergan P, and Nygård M.K. (1989). Nonlinear shell analysis using free formulation finite elements. *Finite element methods for nonlinear problems*, Springer Verlag, 317-338.

Brank B., Peric D., and Dajmanic F.B. (1997). On large deformations of thin elasto-plastic shells: implementation of a finite rotation model for quadrilateral shelll element. *Int. J. Numerical Meth. Engng*, vol.40, 689-726.

Crisfield M. and Peng X. (1992). Efficient nonlinear shell formulation with large rotations and palsticity. *COMPLAS 3*, Barcelona, 1979-1997.

Haugen B. (1994). *Buckling and stability problems for thin shell structures using high performance finite elements*. Ph-D. Thesis, Univ. Colorado, Dept. Aerospace Engng.

Ibrahimbegovic A., and Frey F. (1992). Stress resultant elasto-plastic analysis of plates and shallow shells. *COMPLAS 3*, Barcelona, 2047-2059.

Matthies H.G. (1989). A decomposition method for integration of the elastic-plastic rate problem. *Int.J.Numerical Meth.Engng*, vol.28, 1-11.

Militello C., and Felippa C.A. (1991). The first ANDES elements: 9-DOF plate bending triangles. *Comp.Meth.Aplied Mech. Engng*. , vol.93, 217-246.

Simo J.C., and Kennedy J.G. (1992). On a stress resultant geomtrically exact shell model- Part V-Nonlinear plasticity: formulation and integration algorithms. *Comp. Meth. Applied Mech. Engng*, vol.96, 133-171.

Søreide T. (1977). *Collapse behaviour of stiffened plates using alternative finite element formulations*. Dr. Thesis, Div. Struct. Engng, NTH.