SHELL EQUATIONS

Stress resultants per unit length acting on an element of a cylindrical shell are shown in figure 1.

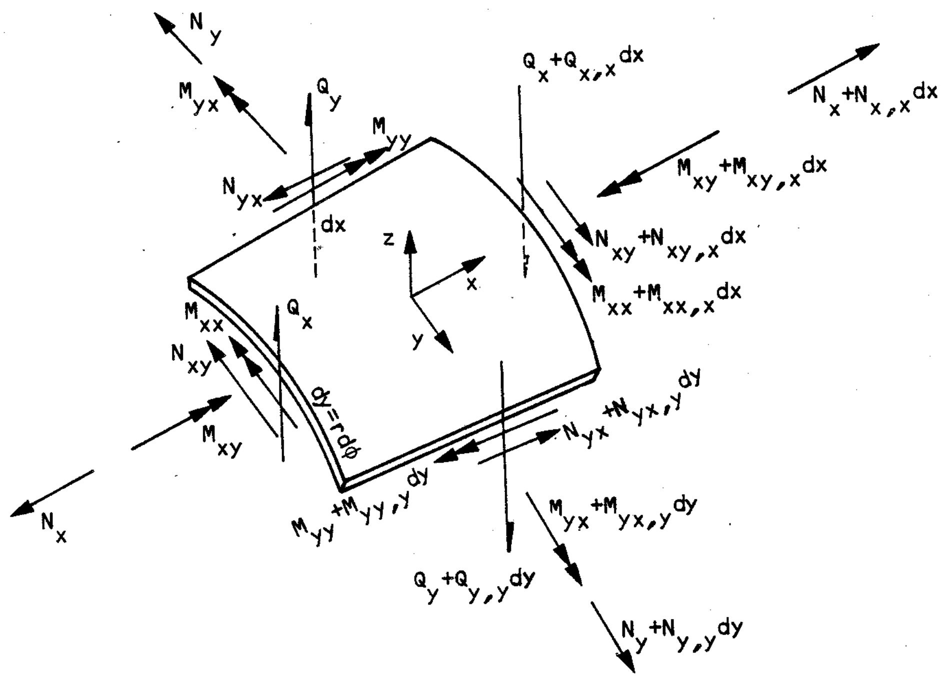

Figure 1   Forces acting on a shell element

Equilibrium gives :

$$N_{x,x} + N_{yx,y} + X = 0 \qquad (1)$$

$$N_{y,y} + N_{xy,x} + \frac{1}{R} Q_y + Y = 0 \qquad (2)$$

$$Q_{y,y} + Q_{x,x} + \frac{1}{R} N_y - Z = 0 \qquad (3)$$

$$M_{yy,y} + M_{xy,x} - Q_y = 0 \qquad (4)$$

$$M_{xx,x} + M_{yx,y} - Q_x = 0 \qquad (5)$$

$$M_{yx} + R\,N_{xy} - R\,N_{yx} = 0 \qquad (6)$$

The following approximations:

$$M_{yx} \cdot\cdot\ M_{xy} \qquad (7)$$

$$N_{yx} \cdot\cdot\ N_{xy} \qquad (8)$$

Gives the following three equations of equilibrium in the X-, Y- and Z-directions:

$$N_{x,x} + N_{xy,y} + X = 0 \qquad (9)$$

$$N_{y,y} + N_{xy,x} + \frac{1}{R} Q_y + Y = 0 \qquad (10)$$

$$M_{xx,xx} + 2\,M_{xy,xy} + M_{yy,yy} + \frac{1}{R} N_y - Z = 0 \qquad (11)$$

The relationship between stress resultants and the shell displacements u, v and w:

$$M_{xx} = D_0 (\ w_{,xx} + o\ w_{,yy}\ ) \qquad (12)$$

$$M_{yy} = D_0 (\ w_{,yy} + o\ w_{,xx}\ )$$

(13)

$$M_{xy} = D_0 (\ 1 - o\ )\ w_{,xy} \qquad (14)$$

$$N_y = \frac{E\,t}{(\ 1-o^2\ )} (\ v_{,y} + \frac{w}{R} + o\ u_{,x}\ ) \qquad (15)$$

$$N_x = \frac{E\,t}{(\ 1-o^2\ )} (\ u_{,x} + o\,(v_{,y} + \frac{w}{R}\ )\ ) \qquad (16)$$

$$N_{xy} = \frac{E\,t}{2\,(1+o\,)} (\ v_{,x} + u_{,y}\ ) \qquad (17)$$

The shell stiffness is

$$D_0 = \frac{E\,t^3}{12\,(1-o^2\,)} \qquad (18)$$

Equilibrium equations expressed by the displacements u, v and w:

X -Direction : (19)

$$\frac{E\,t}{(1-o^2\,)} (\ u_{,xx} + o\ v_{,xy} + \frac{o}{R}\ w_{,x}\ ) + \frac{E\,t}{2\,(1+o\,)} (\ v_{,xy} + u_{,yy}\ ) + X = 0$$

Y -Direction : (20)

$$\frac{E\,t}{(1-o^2\,)} (\ v_{,yy} + \frac{1}{R}\ w_{,y} + o\ u_{,xy}\ ) + \frac{E\,t}{2\,(1+o\,)} (\ v_{,xx} + u_{,xy}\ ) + \frac{D_0}{R} (\ w_{,yyy} + w_{,xxy}\ ) + Y = 0$$

Z -Direction : (21)

$$D_0 (\ w_{,xxxx} + 2\ w_{,xxyy} + w_{,yyyy}\ ) + \frac{E\,t}{R\,(1-o^2\,)} (\ v_{,y} + \frac{w}{R} + o\ u_{,x}\ ) - Z = 0$$

X -Direction Rearranged: (22)

$$\left(\frac{E\,t}{1-o^{2}}\,u_{,xx}+\frac{E\,t}{2\,(1+o\,)}\,u_{,yy}\right)+\left(\frac{E\,t\,o}{1-o^{2}}+\frac{E\,t}{2\,(1+o\,)}\right)v_{,xy}+\left(\frac{E\,t\,o}{R\,(1-o^{2})}\,w_{,x}\right)+X=0$$

Y -Direction Rearranged: (23)

$$\left(\frac{E\,t\,o}{(1-o^{2})}+\frac{E\,t}{2\,(1+o)}\right)u_{,xy}+\left(\frac{E\,t}{(1-o^{2})}\,v_{,yy}+\frac{E\,t}{2\,(1+o\,)}\,v_{,xx}\right)+\frac{D_0}{R}\left(w_{,yyy}+w_{,xxy}+\frac{12}{t^{2}}\,w_{,y}\right)+Y=0$$

Z -Direction Rearranged: (24)

$$\left(\frac{E\,t\,o}{R\,(1-o^{2}\,)}\right)u_{,x}+\left(\frac{E\,t}{R\,(1-o^{2}\,)}\right)v_{,y}+D_0\left(w_{,xxxx}+2\,w_{,xxyy}+w_{,yyyy}\right)+\left(\frac{E\,t}{R^{2}\,(1-o^{2}\,)}\right)w-Z=0$$

$$w = A_{mn} \sin(m\frac{\theta x}{L}) \cos(n\gamma)$$

$$w_{,x} = (\frac{m\theta}{L}) A_{mn} \cos(m\frac{\theta x}{L}) \cos(n\gamma)$$

$$w_{,xx} = -(\frac{m\theta}{L})^2 A_{mn} \sin(m\frac{\theta x}{L}) \cos(n\gamma)$$

$$w_{,xxx} = -(\frac{m\theta}{L})^3 A_{mn} \cos(m\frac{\theta x}{L}) \cos(n\gamma)$$

$$w_{,xxxx} = (\frac{m\theta}{L})^4 A_{mn} \sin(m\frac{\theta x}{L}) \cos(n\gamma)$$

$$w_{,xxy} = (\frac{n}{R})(\frac{m\theta}{L})^2 A_{mn} \sin(m\frac{\theta x}{L}) \sin(n\gamma)$$

$$w_{,xxyy} = (\frac{n}{R})^2 (\frac{m\theta}{L})^2 A_{mn} \sin(m\frac{\theta x}{L}) \cos(n\gamma)$$

$$w_{,y} = -(\frac{n}{R}) A_{mn} \sin(m\frac{\theta x}{L}) \sin(n\gamma)$$

$$w_{,yyy} = (\frac{n}{R})^3 A_{mn} \sin(m\frac{\theta x}{L}) \sin(n\gamma)$$

$$w_{,yyyy} = (\frac{n}{R})^4 A_{mn} \sin(m\frac{\theta x}{L}) \cos(n\gamma)$$

$$v = B_{mn} \sin(m\frac{\theta x}{L}) \sin(n\gamma)$$

$$v_{,xx} = -(\frac{m\theta}{L})^2 B_{mn} \sin(m\frac{\theta x}{L}) \sin(n\gamma)$$

$$v_{,xy} = (\frac{n}{R})(\frac{m\theta}{L}) B_{mn} \cos(m\frac{\theta x}{L}) \cos(n\gamma)$$

$$v_{,y} = (\frac{n}{R}) B_{mn} \sin(m\frac{\theta x}{L}) \cos(n\gamma)$$

$$v_{,yy} = -(\frac{n}{R})^2 B_{mn} \sin(m\frac{\theta x}{L}) \sin(n\gamma)$$

$$u = C_{mn} \cos(m \frac{2\theta x}{L}) \cos(n\gamma)$$

$$u_{,x} = -(m \frac{2\theta}{L}) C_{mn} \sin(m \frac{2\theta x}{L}) \cos(n\gamma)$$

$$u_{,xx} = -(m \frac{2\theta}{L})^2 C_{mn} \cos(m \frac{2\theta x}{L}) \cos(n\gamma)$$

$$u_{,xy} = (\frac{n}{R})(m \frac{2\theta}{L}) C_{mn} \sin(m \frac{2\theta x}{L}) \sin(n\gamma)$$

$$u_{,yy} = -(\frac{n}{R})^2 C_{mn} \cos(m \frac{2\theta x}{L}) \cos(n\gamma)$$

$$I_X 11^s = \int_0^L \sin(m \frac{\theta x}{L}) \sin(k \frac{\theta x}{L}) dx = \frac{L}{2}, \quad for\ m = k$$

$$I_X 21^s = \int_0^L \sin(m \frac{2\theta x}{L}) \sin(k \frac{\theta x}{L}) dx = \frac{L}{2}, \quad for\ m = 2k$$

$$I_X 22^s = \int_0^L \sin(m \frac{2\theta x}{L}) \sin(k \frac{2\theta x}{L}) dx = \frac{L}{2}, \quad for\ m = k$$

$$I_X 11^c = \int_0^L \cos(m \frac{\theta x}{L}) \cos(k \frac{\theta x}{L}) dx = \frac{L}{2}, \quad for\ m = k$$

$$I_X 21^c = \int_0^L \cos(m \frac{2\theta x}{L}) \cos(k \frac{\theta x}{L}) dx = \frac{L}{2}, \quad for\ m = 2k$$

$$I_X 22^c = \int_0^L \cos(m \frac{2\theta x}{L}) \cos(k \frac{2\theta x}{L}) dx = \frac{L}{2}, \quad for\ m = k$$

$$I_Y 11^s = \int_{-\theta}^{\theta} \sin(n\gamma) \sin(l\gamma) R\, d\gamma = R\theta \quad for\ n = l$$

$$I_Y 11^c = \int_{-\theta}^{\theta} \cos(n\gamma) \cos(l\gamma) R\, d\gamma = R\theta \quad for\ n = l$$

$$f_X 1 C_{mn} + f_X 2 B_{mn} + f_X 3 A_{mn} = - X \cos(m \frac{2\theta x_0}{L}) \cos(n \gamma_0)$$

$$f_Y 1 C_{mn} + f_Y 2 B_{mn} + f_Y 3 A_{mn} = - Y \sin(m \frac{\theta x_0}{L}) \sin(n \gamma_0)$$

$$f_Z 1 C_{mn} + f_Z 2 B_{mn} + f_Z 3 A_{mn} = Z \sin(m \frac{\theta x_0}{L}) \cos(n \gamma_0)$$

Virtual displacement in X-direction:

$$\tilde{u}(x,\gamma) = \sum_k \sum_l \cos(k \frac{2\theta x}{L}) \cos(l \gamma)$$

Gives the following coefficients:

$$f_X 1 = [\frac{E t}{1 - o^2} (-(\frac{2\theta m}{L})^2) + \frac{E t}{2(1+o)} (-(\frac{n}{R})^2)] I_X 22^c I_Y 11^c$$

$$f_X 2 = (\frac{E t o}{1 - o^2} + \frac{E t}{2(1+o)}) (\frac{m\theta}{L}) (\frac{n}{R}) I_X 21^c I_Y 11^c$$

$$f_X 3 = (\frac{E t o}{R(1 - o^2)}) (\frac{m\theta}{L}) I_X 21^c I_Y 11^c$$

Virtual displacement in Y-direction:

$$\tilde{v}(x,\gamma) = \sum_k \sum_l \sin(k \frac{\theta x}{L}) \sin(l \gamma)$$

Gives the following coefficients:

$$f_Y 1 = (\frac{E t o}{1 - o^2} + \frac{E t}{2(1+o)}) (\frac{2 m\theta}{L}) (\frac{n}{R}) I_X 21^s I_Y 11^s$$

$$f_Y 2 = [\frac{E t}{1 - o^2} (-(\frac{n}{R})^2) + \frac{E t}{2(1+o)} (-(\frac{m\theta}{L})^2)] I_X 11^s I_Y 11^s$$

$$f_Y 3 = \frac{D_0}{R} [(\frac{n}{R})^3 + (\frac{n}{R}) (\frac{m\theta}{L})^2 + \frac{12}{t^2} (- \frac{n}{R})] I_X 11^s I_Y 11^s$$

Virtual displacement in Z-direction:

$$\tilde{w}(x,\gamma) = \sum_k \sum_l \sin(k \frac{\theta x}{L}) \cos(l \gamma)$$

Gives the following coefficients:

$$f_Z 1 = (\frac{E t o}{R(1 - o^2)}) (- \frac{2 m\theta}{L}) I_X 21^s I_Y 11^c$$

$$f_Z 2 = \frac{E\,t}{R(1 - o^2)} \left(\frac{n}{R} \right) I_X 11^s\, I_Y 11^c$$

$$f_Z 3 = D_0 \left[\left(\frac{m\theta}{L} \right)^4 + 2 \left(\frac{n}{R} \right)^2 \left(\frac{m\theta}{L} \right)^2 + \left(\frac{n}{R} \right)^4 + \frac{12}{R^2 t^2} \right] I_X 11^s\, I_Y 11^c$$

TANGENTIAL LOAD:

$$w = A_{mn} \sin(m\frac{\theta x}{L}) \sin(n\gamma)$$

$$w_{,x} = (\frac{m\theta}{L}) A_{mn} \cos(m\frac{\theta x}{L}) \sin(n\gamma)$$

$$w_{,xx} = -(\frac{m\theta}{L})^2 A_{mn} \sin(m\frac{\theta x}{L}) \sin(n\gamma)$$

$$w_{,xxx} = -(\frac{m\theta}{L})^3 A_{mn} \cos(m\frac{\theta x}{L}) \sin(n\gamma)$$

$$w_{,xxxx} = (\frac{m\theta}{L})^4 A_{mn} \sin(m\frac{\theta x}{L}) \sin(n\gamma)$$

$$w_{,xxy} = -(\frac{n}{R})(\frac{m\theta}{L})^2 A_{mn} \sin(m\frac{\theta x}{L}) \cos(n\gamma)$$

$$w_{,xxyy} = (\frac{n}{R})^2(\frac{m\theta}{L})^2 A_{mn} \sin(m\frac{\theta x}{L}) \sin(n\gamma)$$

$$w_{,y} = (\frac{n}{R}) A_{mn} \sin(m\frac{\theta x}{L}) \cos(n\gamma)$$

$$w_{,yyy} = -(\frac{n}{R})^3 A_{mn} \sin(m\frac{\theta x}{L}) \cos(n\gamma)$$

$$w_{,yyyy} = (\frac{n}{R})^4 A_{mn} \sin(m\frac{\theta x}{L}) \sin(n\gamma)$$

$$v = B_{mn} \sin(m\frac{\theta x}{L}) \cos(n\gamma)$$

$$v_{,xx} = -(\frac{m\theta}{L})^2 B_{mn} \sin(m\frac{\theta x}{L}) \cos(n\gamma)$$

$$v_{,xy} = -(\frac{n}{R})(\frac{m\theta}{L}) B_{mn} \cos(m\frac{\theta x}{L}) \sin(n\gamma)$$

$$v_{,y} = -(\frac{n}{R}) B_{mn} \sin(m\frac{\theta x}{L}) \sin(n\gamma)$$

$$v_{,yy} = -(\frac{n}{R})^2 B_{mn} \sin(m\frac{\theta x}{L}) \cos(n\gamma)$$

$$u = C_{mn} \cos(m\frac{\theta x}{L}) \sin(n\gamma)$$

$$u_{,x} = -(m\frac{\theta}{L}) C_{mn} \sin(m\frac{\theta x}{L}) \sin(n\gamma)$$

$$u_{,xx} = -(m\frac{\theta}{L})^2 C_{mn} \cos(m\frac{\theta x}{L}) \sin(n\gamma)$$

$$u_{,xy} = -(\frac{n}{R})(m\frac{\theta}{L}) C_{mn} \sin(m\frac{\theta x}{L}) \cos(n\gamma)$$

$$u_{,yy} = -(\frac{n}{R})^2 C_{mn} \cos(m\frac{\theta x}{L}) \sin(n\gamma)$$

$$I_X 11^s = \int_0^L \sin(m\frac{\theta x}{L}) \sin(k\frac{\theta x}{L})\, dx = \frac{L}{2}, \quad for\ m = k$$

$$I_X 21^s = \int_0^L \sin(m\frac{2\theta x}{L}) \sin(k\frac{\theta x}{L})\, dx = \frac{L}{2}, \quad for\ m = 2k$$

$$I_X 22^s = \int_0^L \sin(m\frac{2\theta x}{L}) \sin(k\frac{2\theta x}{L})\, dx = \frac{L}{2}, \quad for\ m = k$$

$$I_X 11^c = \int_0^L \cos(m\frac{\theta x}{L}) \cos(k\frac{\theta x}{L})\, dx = \frac{L}{2}, \quad for\ m = k$$

$$I_X 21^c = \int_0^L \cos(m\frac{2\theta x}{L}) \cos(k\frac{\theta x}{L})\, dx = \frac{L}{2}, \quad for\ m = 2k$$

$$I_X 22^c = \int_0^L \cos(m\frac{2\theta x}{L}) \cos(k\frac{2\theta x}{L})\, dx = \frac{L}{2}, \quad for\ m = k$$

$$I_Y 11^s = \int_{-\theta}^{\theta} \sin(n\gamma) \sin(l\gamma)\, R\, d\gamma = R\theta \quad for\ n = l$$

$$I_Y 11^c = \int_{-\theta}^{\theta} \cos(n\gamma) \cos(l\gamma)\, R\, d\gamma = R\theta \quad for\ n = l$$

$$g_X 1\, C_{mn} + g_X 2\, B_{mn} + g_X 3\, A_{mn} = -X \cos(m \frac{\theta\, x_0}{L}) \sin(n \gamma_0)$$

$$g_Y 1\, C_{mn} + g_Y 2\, B_{mn} + g_Y 3\, A_{mn} = -Y \sin(m \frac{\theta\, x_0}{L}) \sin(n \gamma_0)$$

$$g_Z 1\, C_{mn} + g_Z 2\, B_{mn} + g_Z 3\, A_{mn} = Z \sin(m \frac{\theta\, x_0}{L}) \cos(n \gamma_0)$$

Virtual displacement in X-direction:

$$\tilde{u}(x,\gamma) = \sum_k \sum_l \cos(k \frac{\theta\, x}{L}) \sin(l \gamma)$$

Gives the following coefficients:

$$g_X 1 = [\frac{E\,t}{1-o^2}(-(\frac{m\,\theta}{L})^2) + \frac{E\,t}{2(1+o)}(-(\frac{n}{R})^2)]\, I_X 22^c\, I_Y 11^{ss}$$

$$g_X 2 = -(\frac{E\,t\,o}{1-o^2} + \frac{E\,t}{2(1+o)})(\frac{m\,\theta}{L})(\frac{n}{R})\, I_X 21^c\, I_Y 11^{ss}$$

$$g_X 3 = (\frac{E\,t\,o}{R(1-o^2)})(\frac{m\,\theta}{L})\, I_X 21^c\, I_Y 11^{ss}$$

Virtual displacement in Y-direction:

$$\tilde{v}(x,\gamma) = \sum_k \sum_l \sin(k \frac{\theta\, x}{L}) \cos(l \gamma)$$

Gives the following coefficients:

$$g_Y 1 = -(\frac{E\,t\,o}{1-o^2} + \frac{E\,t}{2(1+o)})(\frac{m\,\theta}{L})(\frac{n}{R})\, I_X 21^s\, I_Y 11^{cc}$$

$$g_Y 2 = [\frac{E\,t}{1-o^2}(-(\frac{n}{R})^2) + \frac{E\,t}{2(1+o)}(-(\frac{m\,\theta}{L})^2)]\, I_X 11^s\, I_Y 11^{cc}$$

$$g_Y 3 = \frac{D_0}{R}[-(\frac{n}{R})^3 - (\frac{n}{R})(\frac{m\,\theta}{L})^2 + \frac{12}{t^2}(\frac{n}{R})]\, I_X 11^s\, I_Y 11^{cc}$$

Virtual displacement in Z-direction:

$$\tilde{w}(x,\gamma) = \sum_k \sum_l \sin(k \frac{\theta\, x}{L}) \sin(l \gamma)$$

Gives the following coefficients:

$$g_Z 1 = \left(\frac{E\, t\, o}{R\,(1-o^2)} \right)\left(-\frac{m\,\theta}{L} \right) I_X\, 21^s\, I_Y\, 11^{ss}$$

$$g_Z 2 = \frac{E\, t}{R\,(1-o^2)}\left(-\frac{n}{R} \right) I_X\, 11^s\, I_Y\, 11^{ss}$$

$$g_Z 3 = D_0 \left[\left(\frac{m\,\theta}{L} \right)^4 + 2\left(\frac{n}{R} \right)^2 \left(\frac{m\,\theta}{L} \right)^2 + \left(\frac{n}{R} \right)^4 + \frac{12}{R^2\, t^2} \right] I_X\, 11^s\, I_Y\, 11^{ss}$$

SHELL FORCES

The stress resultants expressed with the shell displacement functions for u, v and w:

The solution for **radial** load:

$$M_{xx} = D_0\left(-\left(\frac{m\theta}{L}\right)^2 - o\left(\frac{n}{R}\right)^2 \right) A_{mn} \sin\left(\frac{m\theta x}{L}\right)\cos(n\gamma)$$

$$M_{yy} = D_0\left(-o\left(\frac{m\theta}{L}\right)^2 - \left(\frac{n}{R}\right)^2 \right) A_{mn} \sin\left(\frac{m\theta x}{L}\right)\cos(n\gamma)$$

$$M_{xy} = D_0 (1-o)\left(-\frac{m\theta}{L}\right)\left(\frac{n}{R}\right) A_{mn} \cos\left(\frac{m\theta x}{L}\right)\sin(n\gamma)$$

$$N_x = \frac{E t}{1-o^2}\left(\frac{o}{R} A_{mn} + o\frac{n}{R} B_{mn} + \left(-\frac{m\theta}{L}\right) C_{mn} \right)\sin\left(\frac{m\theta x}{L}\right)\cos(n\gamma)$$

$$N_y = \frac{E t}{1-o^2}\left(\frac{1}{R} A_{mn} + \frac{n}{R} B_{mn} + o\left(-\frac{m\theta}{L}\right) C_{mn} \right)\sin\left(\frac{m\theta x}{L}\right)\cos(n\gamma)$$

$$N_{xy} = \frac{E t}{2(1+o)}\left(\frac{m\theta}{L} B_{mn} + \left(-\frac{n}{R}\right) C_{mn} \right)\cos\left(\frac{m\theta x}{L}\right)\sin(n\gamma)$$

The solution for **tangential** load:

$$M_{xx} = D_0 \left(-\left(\frac{m\theta}{L} \right)^2 - o \left(\frac{n}{R} \right)^2 \right) A_{mn} \sin\left(\frac{m\theta x}{L} \right) \sin(n\gamma)$$

$$M_{yy} = D_0 \left(-o \left(\frac{m\theta}{L} \right)^2 - \left(\frac{n}{R} \right)^2 \right) A_{mn} \sin\left(\frac{m\theta x}{L} \right) \sin(n\gamma)$$

$$M_{xy} = D_0 \, (1 - o) \left(+ \frac{m\theta}{L} \right) \left(\frac{n}{R} \right) A_{mn} \cos\left(\frac{m\theta x}{L} \right) \cos(n\gamma)$$

$$N_x = \frac{E\,t}{1 - o^2} \left(\frac{o}{R} A_{mn} - o \frac{n}{R} B_{mn} + \left(- \frac{m\theta}{L} \right) C_{mn} \right) \sin\left(\frac{m\theta x}{L} \right) \sin(n\gamma)$$

$$N_y = \frac{E\,t}{1 - o^2} \left(\frac{1}{R} A_{mn} - \frac{n}{R} B_{mn} + o\left(- \frac{m\theta}{L} \right) C_{mn} \right) \sin\left(\frac{m\theta x}{L} \right) \sin(n\gamma)$$

$$N_{xy} = \frac{E\,t}{2(1 + o)} \left(\frac{m\theta}{L} B_{mn} + \left(+ \frac{n}{R} \right) C_{mn} \right) \cos\left(\frac{m\theta x}{L} \right) \cos(n\gamma)$$